The known finite Minkowski planes — a characterization in terms of Klein–Kroll types

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13 December 2018

What is a Minkowski plane?

A *(B^*)-geometry* or *hyperbola structue* $\mathcal{M} = (P, \mathcal{C}, \mathcal{G}_1 \cup \mathcal{G}_2)$ is an incidence structure consisting of a point set P, a circle set $\mathcal{C}$, elements of which are subsets of P with at least three points, and two different partitions $\mathcal{G}_1$ and $\mathcal{G}_2$ of P, whose members are called generators of $\mathcal{M}$, such that the following three axioms are satisfied:

(G) Each generator in $\mathcal{G}_1$ intersects each generator in $\mathcal{G}_2$ in a unique point.

(C) Each circle intersects each generator in precisely one point.

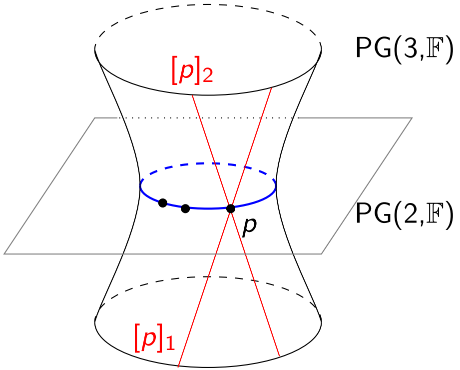
(J) Three points no two of which are on the same generator are joined by a unique circle.

A *Minkowski plane* is a (B^*)-geometry that also satisfies the axiom

(T) The circles which touch a fixed circle C at $p \in C$ partition $P \setminus [p]$.

Models of Minkowski planes

The *miquelian Minkowski plane* over a field $\mathbb{F}$ is obtained as the geometry of non-trivial plane sections of a ruled quadric in 3-dimensional projective space over $\mathbb{F}$.



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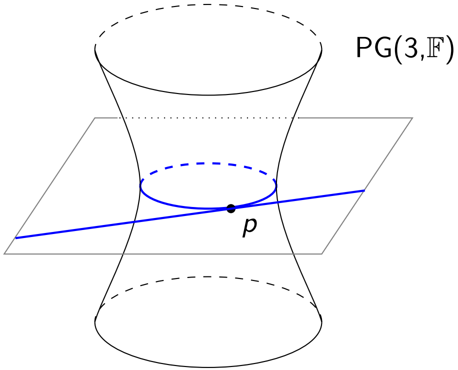
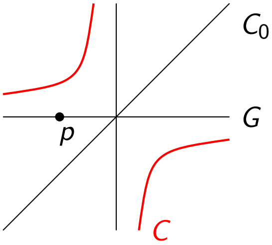


An equivalent description

A (B^*)-geometry corresponds to a sharply 3-transitive **set** Σ of permutations on a generator G; circles are the graphs of permutations in Σ.



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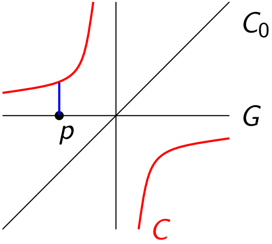

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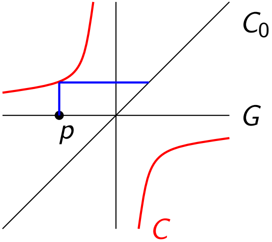

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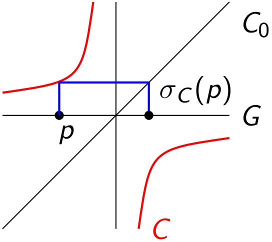

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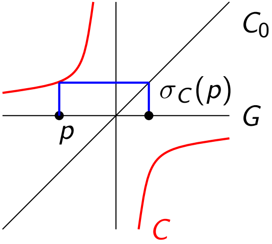

The miquelian Minkowski plane over the field $\mathbb{F}$ corresponds to the group PGL(2,$\mathbb{F}$) of linear fractional maps acting on $\mathbb{F} \cup \{\infty\}$.

The known finite Minkowski planes

All known finite Minkowski planes are of the form $\mathcal{M}(q, \alpha)$ obtained from sharply 3-transitive sets

$$G(q, \alpha) = \mathrm{PSL}(2, q) \cup (\mathrm{PGL}(2, q) \setminus \mathrm{PSL}(2, q))\alpha$$

where q is a prime power and α is an automorphism of $\mathrm{GF}(q)$.

Circles are the graphs of permutations in $G(q, \alpha)$ on $\mathrm{GF}(q) \cup \{\infty\}$.

- The miquelian Minkowski planes are obtained when $\alpha =$id.
- $G(q, \alpha)$ is a group if and only if α has order at most 2.
- A finite hyperlola structure is a Minkowski plane.

A finite Minkowski plane has order n if each generator and circle has precisely $n + 1$ points.

Derived incidence structures and consequences

The derived incidence structure $\mathcal{M}_p$ at a point p of a Minkowski plane $\mathcal{M}$ is an affine plane.
A circle C not passing through p induces an oval in the projective extension of $\mathcal{M}_p$ by removing the points $C \cap [p]$ and adding the points at infinity of lines that come from generators of $\mathcal{M}$.

Theorem

- *A finite Minkowski plane of even order is miquelian. (Heise 1974)*
- *A finite Minkowski plane of odd order with a Desarguesian derivation is miquelian. (Chen, Kaerlein 1973, Payne, Thas 1976)*
- *A finite Minkowski plane of order at most 8 is miquelian.*
- *There are precisely two finite Minkowski planes of order 9, up to isomorphism. These planes correspond to the two sharply 3-transitive groups of degree 10. (S. 1992)*

G-translations

- An *automorphism* of a Minkowski plane $\mathcal{M}$ is a permutation of the point set such that generators are mapped to generators and circles are mapped to circles.
- A *G-translation* of $\mathcal{M}$ is an automorphism of $\mathcal{M}$ that either fixes precisely the points of the generator G or is the identity; it induces a translation in the derived affine plane of $\mathcal{M}$ at any point of G.
- A group Γ of automorphisms of $\mathcal{M}$ is said to be *G-transitive* if Γ contains a subgroup of G-translations that acts transitively on each circle minus its point of intersection with G.

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M. Klein and H.-J. Kroll [1989] considered the set $\mathcal{Z}(\Gamma)$ of all generators G for which a group Γ of automorphisms of $\mathcal{M}$ is G-transitive. They found six types for Γ, labelled A to F.

The six Klein–Kroll types w.r.t. $\mathcal{G}$-translations

Theorem (Klein, Kroll, 1989)

If $\mathcal{Z} = \mathcal{Z}(\Gamma)$ denotes the set of all generators G for which a group Γ of automorphisms of a hyperbola structure is G-transitive, then exactly one of the following statements is valid for $\mathcal{Z}$:

A. $\mathcal{Z} = \emptyset$;

B. $|\mathcal{Z}| = 1$;

C. $\mathcal{Z} = \{[p]_1, [p]_2\}$ *for some point* p;

D. $\mathcal{Z} = \mathcal{G}_1$ *or* $\mathcal{Z} = \mathcal{G}_2$;

E. $\mathcal{Z} = \mathcal{G}_1 \cup \{G_2\}$ *or* $\mathcal{Z} = \mathcal{G}_2 \cup \{G_1\}$ *where* $G_i \in \mathcal{G}_i$;

F. $\mathcal{Z} = \mathcal{G}_1 \cup \mathcal{G}_2$.

There are examples of groups of automorphisms of miquelian Minkowski planes for each of the six types.

The Klein–Kroll type of a Minkowski plane

The *type of a Minkowski plane* $\mathcal{M}$ is the type of the (full) automorphism group of $\mathcal{M}$.

Question: Which types do occur as the type of a (finite) Minkowski plane?

The planes $\mathcal{M}(q, \alpha)$ are of type F. Each map

$$(x, y) \mapsto (\gamma_1(x), \gamma_2(y))$$

where $\gamma_1, \gamma_2 \in \mathrm{PSL}(2, q)$ is an automorphism of $\mathcal{M}(q, \alpha)$. Those automorphisms with $\gamma_i =$id and γ_{3-i} fixing precisely one point are $\mathcal{G}$-translations of $\mathcal{M}(q, \alpha)$.

Type at least D and 2-transitive groups

Lemma

Let $\mathcal{M}$ be a Minkowski plane whose automorphism group is G-transitive for each $G \in \mathcal{G}_1$. Then the group generated by all G-translations for $G \in \mathcal{G}_1$ acts 2-transitively on $\mathcal{G}_1$ and trivially on $\mathcal{G}_2$. Furthermore, the stabilizer of three points no two of which are on the same generator in $\mathcal{G}_1$ is trivial.

Theorem (Feit 1960, Ito, Suzuki 1962)

If Π is a 2-transitive permutation group of even degree $n+1$ such that only the identity fixes more than two points, then one of the following occurs:

1. *Π is sharply 2-transitive (and isomorphic to the group of all permutations $x \mapsto xa + b$, where $a \neq 0$, of a nearfield of order $n+1$).*
2. *$\Pi \cong \mathrm{A\Gamma L}(1, n+1)$ where $n = 2^q - 1$ and q is a prime.*
3. *Π contains $\mathrm{PSL}(2, n)$ as a normal subgroup of index at most 2.*

Type E

Theorem

Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group is G-transitive for each $G \in \mathcal{G}_i$. If the group Δ generated by all G-translations for $G \in \mathcal{G}_i$ is non-solvable, then the order of $\mathcal{M}$ is a prime power q and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.

Theorem

Let $\mathcal{M}$ be a finite Minkowski plane whose automorphism group contains a group of type E. Then the order of $\mathcal{M}$ is a prime power q and $\mathcal{M}$ is isomorphic to a plane $\mathcal{M}(q, \alpha)$.

Corollary

There is no finite Minkowski plane of type E.

There are infinite Minkowski planes of types A, B, C, D and F.

The characterization and a conjecture

Theorem

The Minkowski planes $\mathcal{M}(q, \alpha)$ are precisely the finite Minkowski planes of Klein–Kroll type at least E.

Conjecture

There is no finite Minkowski plane of type D.

The conjecture will follow if the Prime Power Conjecture for finite projective planes and the longstanding conjecture that a projective plane of prime order is desarguesian are both true.